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THE WAVE POTENTIAL OF A CIRCULAR LINE OF SOURCES.

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The fundamental solution of the wave equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\Delta\phi, \tag{1}$$

obtained by putting $\phi = ue^{int}$, so that u satisfies

$$\Delta u + \kappa^2 u = 0, \tag{2}$$

and representing a source, is $u = e^{ikr}/r$ ($r =$ distance from source). Now we have Hankel's formula for an arbitrary function,

$$F(r) = \int_0^\infty \lambda d\lambda \int_0^\infty \rho d\rho\, F(\rho)\, J(\lambda\rho)\, J(\lambda r), \quad (J = J_0, \text{ Bessel's func.}) \tag{3}$$

and putting

$$F(r) \equiv e^{ikr}/r,$$

$$\frac{e^{ikr}}{r} = \int_0^\infty \lambda d\lambda \int_0^\infty d\rho\, e^{ik\rho}\, J(\lambda\rho)\, J(\lambda r), \tag{4}$$

and since

$$\int_0^\infty d\rho\, e^{ik\rho}\, J(\lambda\rho) = \frac{1}{\sqrt{\lambda^2 - \kappa^2}}, \tag{5}$$

this gives

$$\frac{e^{ikr}}{r} = \int_0^\infty \frac{\lambda d\lambda\; J(\lambda r)}{\sqrt{\lambda^2 - \kappa^2}}. \tag{6}$$

If we now use r in a different sense as

$$r^2 = x^2 + y^2, \quad \text{while} \quad R^2 = x^2 + y^2 + z^2,$$

we easily obtain the solution

$$\frac{e^{tkR}}{R} = \int_0^\infty \frac{e^{-z\sqrt{\lambda^2 - k^2}}\,\lambda J(\lambda r)\,d\lambda}{\sqrt{\lambda^2 - \kappa^2}}. \tag{7}$$

This result was given by Sommerfeld, Ann. der Phys. 1909, Bd. 28, p. 683.

From this I obtain for a *ring* of sources of radius a,

$$u = \int_0^\infty \frac{e^{-z\sqrt{\lambda^2 - k^2}}}{\sqrt{\lambda^2 - \kappa^2}}\,\lambda\, J(\lambda r)\, J(\lambda a)\, d\lambda, \tag{8}$$

which reduces to (7) when $a = 0$, and when $\kappa = 0$, to

$$u = \int_0^\infty e^{-z\lambda}\, J(\lambda r)\, J(\lambda a)\, d\lambda, \tag{9}$$

the well-known expression for the Newtonian potential of a ring; and when $r = 0$, to

$$u(z) = \int_0^\infty \frac{e^{-z\sqrt{\lambda^2 - \kappa^2}}}{\sqrt{\lambda^2 - \kappa^2}}\,\lambda\, J(\lambda a)\, d\lambda, \tag{10}$$

which can be shown to be equal to

$$e^{i\kappa\sqrt{z^2 + a^2}} / \sqrt{z^2 + a^2},$$

the correct expression for points on the axis of symmetry.

Now since

$$J(\lambda r) = \frac{1}{\pi}\int_0^\pi e^{i\lambda r \cos\omega}\, d\omega, \tag{11}$$

we have

$$\begin{aligned} u &= \int_0^\infty \frac{e^{-z\sqrt{\lambda^2 - \kappa^2}}}{\sqrt{\lambda^2 - \kappa^2}}\,\lambda J(\lambda r)\, J(\lambda a)\, d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \int_0^\pi \int_0^\pi d\theta\, d\omega\, e^{i\lambda(r\cos\omega + a\cos b) - z\sqrt{\lambda^2 - \kappa^2}} \cdot \frac{\lambda\, d\lambda}{\sqrt{\lambda^2 - \kappa^2}}. \end{aligned} \tag{12}$$

We will divide this into two parts, the integrals from 0 to κ, and κ to ∞. Putting now

$$\lambda = \kappa \sin\phi, \quad \sqrt{\lambda^2 - \kappa^2} = i\kappa\cos\phi, \quad d\lambda = \kappa\cos\phi\, d\phi,$$

$$A = \int_0^{\kappa} e^{-z\sqrt{\lambda^2-\kappa^2}} \frac{\lambda J(\lambda r)}{\sqrt{\lambda^2-\kappa^2}} J(\lambda a)\, d\lambda$$

$$= \frac{-i\kappa}{\pi^2}\int_0^{\pi}\int_0^{\pi} d\theta\, d\omega \int_0^{\pi/2} e^{i\kappa\,[(r\cos\omega + a\cos\theta)\sin\phi - z\cos\phi]} \cdot \sin\phi\, d\phi, \qquad (13)$$

and developing the exponential,

$$= \frac{-i\kappa}{\pi^2}\int_0^{\pi}\int_0^{\pi} d\theta\, d\omega \int_0^{\pi/2} \sum_0^{\infty} \frac{1}{s!} \{i\kappa\,[(r\cos\omega + a\cos\theta)\sin\phi - z\cos\phi]\}^s \cdot \sin\phi\, d\phi.$$

Developing by the polynomial theorem,

$$= \frac{-i\kappa}{\pi^2}\int_0^{\pi/2} \sin\phi\, d\phi \int_0^{\pi}\int_0^{\pi} d\theta\, d\omega \sum_{s=0}^{s=\infty} \sum_{l,m,n}^{l+m+n=s} \frac{(i\kappa)^s (r\cos\omega\sin\phi)^l (a\cos\theta\sin\phi)^m (-z\cos\phi)^n}{l!\ m!\ n!}.$$

Integrating according to θ and ω, the odd powers vanish, and

$$A = -i\kappa \sum_0^{\infty}\sum_0^{\infty}\sum_0^{\infty} \frac{(i\kappa)^{2(l+m)+n}\, r^{2l}}{(2l)!\ (2m)!\ n!} \frac{1\cdot 3\cdot 5 \ldots (2l-1) a^{2m} 1\cdot 3\cdot 5 (2m-1)(-z)^n}{2\cdot 4\cdot 6 \ldots 2l \cdot 2\cdot 4\cdot 6 \ldots 2m}$$

$$\times \int_0^{\pi/2} \sin\phi^{2(l+m)+1} \cos^n\phi\, d\phi, \qquad (14)$$

$$= -i\kappa \sum_0^{\infty}\sum_0^{\infty}\sum_0^{\infty} (i\kappa)^{2(l+m)+n} \frac{r^{2l} a^{2m}}{(2\cdot 4\cdot 6 \ldots 2l)^2 (2\cdot 4\cdot 6 \ldots 2m)^2}$$

$$\times \frac{(-z)^n (l+m)!\, \Gamma\left(\frac{n+1}{2}\right)}{n!\ 2\Gamma\left(\frac{2(l+m)+n+1}{2} + 1\right)}, \qquad (15)$$

$$A = -i\kappa \sum_0^{\infty}\sum_0^{\infty}\sum_0^{\infty} \frac{(i\kappa)^{2(l+m)+n} (l+m)!\ r^{2l} a^{2m} (-z)^n}{2^{l+m}(l!)^2 (m!)^2 n!\,(n+1)(n+3)\ldots[n+2(l+m)+1]}. \qquad (16)$$

In order to obtain the real part we will proceed differently.

$$B = \int_{\kappa}^{\infty} e^{-z\sqrt{\lambda^2-\kappa^2}} \frac{\lambda}{\sqrt{\lambda^2-\kappa^2}} J(\lambda r)\, J(\lambda a)\, d\lambda, \qquad (17)$$

and putting

$$\lambda^2 - \kappa^2 = \kappa^2 t^2, \qquad \frac{\lambda d\lambda}{\sqrt{\lambda^2 - \kappa^2}} = \kappa dt,$$

$$B = \kappa \int_0^\infty e^{-\kappa z t}\, J(\kappa r \sqrt{1 + t^2})\, J(\kappa a \sqrt{1 + t^2})\, dt, \tag{18}$$

which by means of the formula

$$J(\alpha x)\, J(\beta x) = \sum_0^\infty \sum_0^\infty \frac{(-1)^s\, \alpha^{2(s-p)}\, \beta^{2p} \left(\frac{x}{2}\right)^{2s}}{(p\,!)^2\, [(s-p)\,!]^2} \tag{19}$$

becomes

$$B = \kappa \int_0^\infty \sum_{s=0}^{\infty} \sum_{p=0}^{p=s} e^{-\kappa z t} \frac{(-1)^s\, (\kappa r)^{2(s-p)}\, (\kappa a)^{2p} (1 + t^2)^s\, dt}{2^{2s}\, [p\,!\, (s-p)\,!]^2}$$

$$= \kappa \sum_{s=0}^{s=\infty} \sum_{p=0}^{p=s} \sum_{q=0}^{q=s} \frac{(-1)^s\, (\kappa r)^{2(s-p)}\, (\kappa a)^{2p}}{2^{2s}\, [p\,!\, (s-p)\,!]^2} \frac{s\,!}{q\,!\, (s-q)\,!} \int_0^\infty t^{2q}\, e^{-\kappa z t}\, dt$$

$$= \kappa \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} \frac{(-1)^{l+m}\, \kappa^{2(l+m-n)-1}\, r^{2l}\, a^{2m}\, (l+m)\,!\, (2n)\,!}{2^{2(l+m)}\, (l\,!\, m\,!)^2\, n\,!\, (l+m-n)\,!\, z^{2n+1}}. \tag{20}$$

$$u = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=\infty} \frac{(-1)^{l+m}\, \kappa^{2(l+m)}\, (l+m)\,!\, r^{2l}\, a^{2m}}{2^{l+m}\, (l\,!\, m\,!)^2} \sum_{n=0}^{n=\infty}$$

$$\left(\frac{-(i\kappa)^{n+1}\, (-z)^n}{n\,!\, (n+1)(n+3) \ldots [n + 2(l+m) + 1]} + \frac{(2n)\,!}{2^{l+m}\, n\,!\, (l+m-n)\,!\, z^{2n+1}} \right).$$